New Cores for Δ-Edge-Colorable Graphs

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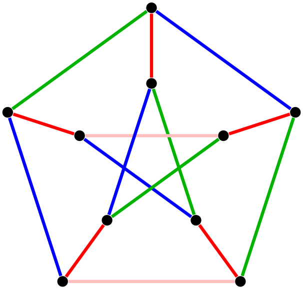
Edge Coloring, Edge Chromatic Number



- A proper edge coloring of a graph assigns a color to each edge so that edges sharing an endpoint (including parallel edges) get different colors.
- The edge chromatic number of G, written $\chi'(G)$, is smallest number of colors in a proper edge coloring.
- $\chi'(\text{Pete}) = 4$.

Vizing's Theorem for Simple Graphs

Theorem (Vizing 1964)

If G is a simple graph, then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Theorem (Holyer 1981)

It is NP-hard to determine, for a simple graph G, whether $\chi'(G) = \Delta(G)$ or whether $\chi'(G) = \Delta(G) + 1$.

Two approaches to continue thinking about edge coloring:

- Look for sufficient conditions. Find a general property P such that if simple G has property P, then $\chi'(G) = \Delta(G)$.
- Extend to multigraphs.

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Fournier's Theorem, Hoffman–Rodger Theorem

Definition

The core of a simple graph G is the subgraph induced by its vertices of degree $\Delta(G)$.

Theorem (Fournier 1973)

Let G be a simple graph. If the core of G is a forest, then $\chi'(G) = \Delta(G)$.

Fourier's Theorem, Hoffman–Rodger Theorem

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Let G be a simple graph. If the core of G is a forest, then $\chi'(G) = \Delta(G)$.

Hoffman–Rodger (1988) defined a B-queue to be a particular vertex ordering of a graph (details later).

Theorem (Hoffman–Rodger 1988)

Let G be a simple graph, and let H be its core. If H admits a "full B-queue", then $\chi'(G) = \Delta(G)$.

Hoffman–Rodger also showed: H is a forest $\implies$ H has a full B-queue.

Fan, cfan, and B-Queues

- Scheide and Stiebitz (2010) introduced a parameter called the Fan number of G, written $\text{Fan}(G)$, and showed that $\chi'(G) \leq \text{Fan}(G)$ for all G. (Details later.)

Fan, cfan, and B-Queues

- Scheide and Stiebitz (2010) introduced a parameter called the Fan number of G, written $\mathrm{Fan}(G)$, and showed that $\chi'(G) \leq \mathrm{Fan}(G)$ for all G. (Details later.)
- $\mathrm{Fan}(G)$ is polynomially computable. Many upper bounds on $\chi'(G)$, including Vizing's Theorem, are actually upper bounds on $\mathrm{Fan}(G)$.
- We define a variant of this parameter, the cfan number, and prove:

Theorem (McDonald–P.)

Let G be a simple graph, and let H be its core. If $\mathrm{cfan}(H) \leq 0$, then $\chi'(G) = \mathrm{Fan}(G) = \Delta(G)$.

- We show:

$$H \text{ is a forest} \implies H \text{ has a full } B\text{-queue} \implies \mathrm{cfan}(H) \leq 0.$$

- However, there are graphs with $\mathrm{cfan}(H) \leq 0$ which do not have a full B-queue.

Vizing's Fan Inequality

If xy is an edge in multigraph J, then $J - xy$ is the subgraph with one fewer copy of xy.

Theorem (Vizing 1964)

Let G be a multigraph, let $k \geq \Delta(G)$, and let $J \subset G$.
Suppose that $J - xy$ is k-edge-colorable for some $xy \in E(J)$.
Either J is k-edge-colorable, or there is a vertex set $Z \subset N_J(x)$ such that:

1. $|Z| \geq 2$,
2. $y \in Z$, *and*
3. $\sum_{z \in Z} (d_J(z) + \mu_J(x,z) - k) \geq 2$.

Conversely: if J is a minimal non-k-edge-colorable subgraph of G, then for every $xy \in E(J)$, such a set Z must exist.

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Observation

If $J - xy$ is k-edge colorable and $d(x) + d(y) - \mu_J(x, y) \leq k$, then J is k-edge-colorable. (Edge xy only sees at most $k - 1$ colors.)

Definition of Fan Number (Scheide–Stiebitz 2010)

Let G be a multigraph. For $J \subset G$ and $xy \in E(J)$, the fan degree of the pair (x, y), written $\deg_J(x, y)$, is the smallest nonnegative k such that either:

1. $d_J(x) + d_J(y) - \mu_J(x, y) \leq k$, or
2. $\sum_{z \in Z}(d_J(z) + \mu_J(x, z) - k) \leq 1$ for all $Z \subset N_J(x)$ with $y \in Z$ and $|Z| \geq 2$.

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The fan number $\text{fan}(G)$ of the graph G is defined by

$$\text{fan}(G) = \max_{\substack{J \subset G \\ E(J) \neq \emptyset}} \min\{\deg_J(x, y) \colon xy \in E(J)\}.$$

The Fan number $\text{Fan}(G)$ is then given by $\text{Fan}(G) = \max\{\text{fan}(G), \Delta(G)\}$.

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Theorem (Scheide–Stiebitz 2010)

For every multigraph G, $\chi'(G) \leq \text{Fan}(G)$.

Idea: by previous slide, if J is a minimal non-k-edge-colorable subgraph for $k \geq \Delta(G)$, then for every pair (x, y), we must have $\deg_J(x, y) > k$.

Definition of cfan

Definition (Scheide–Stiebitz 2010)

$$\mathrm{fan}(G) = \max_{\substack{J \subset G \\ E(J) \neq \emptyset}} \min\{\deg_J(x,y)\colon\ xy \in E(J)\}$$

For $K \subset H$, and $xy \in E(K)$ the cfan degree of the pair (x, y), written $\mathrm{cdeg}_K(x, y)$, is the smallest nonnegative ℓ such that:

$\sum_{z \in Z} (d_K(z) - d_H(z) + \mu_K(x,z) - \ell) \leq 1$ for all $Z \subset N_K(x)$ with $y \in Z$.

The cfan number of H is then defined by

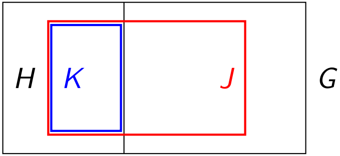
$$\mathrm{cfan}(H) = \max_{\substack{K \subset H \\ E(K) \neq \emptyset}} \min\{\mathrm{cdeg}_{H,K}(x,y)\colon\ xy \in E(K)\}.$$

If $E(H) = \emptyset$, then we put $\mathrm{cfan}(H) = 0$.

Theorem (McDonald–P.)

Let G be a simple graph, and let H be its core. If $\mathrm{cfan}(H) \leq 0$, then $\mathrm{fan}(G) \leq \Delta(G)$. Thus $\chi'(G) \leq \mathrm{Fan}(G) = \Delta(G)$.

Proof Sketch: $\mathrm{cfan}(H) \le 0 \implies \mathrm{fan}(G) \le \Delta(G)$



To show $\mathrm{fan}(G) \le \Delta(G)$, take any nonempty subgraph $J \subset V(G)$.
Need: a pair (x, y) with $xy \in E(J)$ such that $\deg_J(x, y) \le \Delta(G)$.
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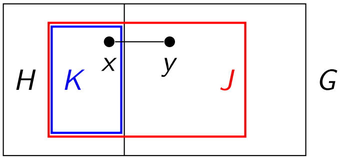

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Easy Case: K does not contain an edge. Let (x, y) be any pair with $xy \in E(J)$, taking $x \in K$ if possible.

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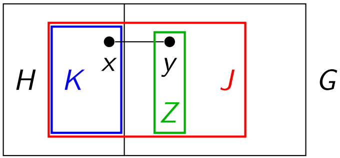

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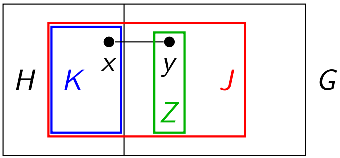

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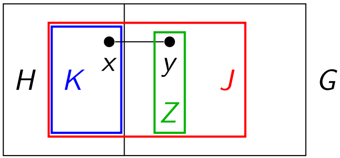

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- Any $z \in N_J(x)$ is outside core, hence $d_G(z) + 1 \le \Delta(G)$.
- Therefore,

$$\sum_{z \in Z} (d_J(z) + 1 - \Delta(G)) \le \sum_{z \in Z} 0 = 0.$$

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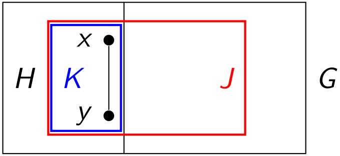

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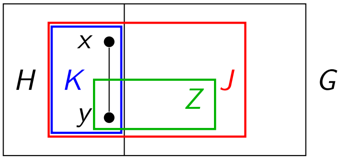

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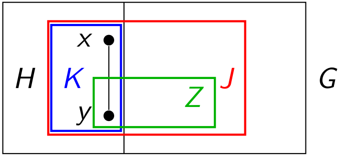

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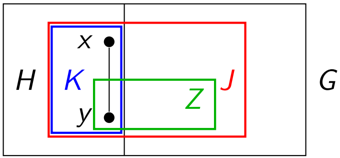

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- For vertices $z \in Z \cap V(K)$, we have

$$d_J(z) + 1 \le \Delta(H) + (d_K(z) - d_H(z)) + 1,$$

  with an upper bound on $\sum_{z \in Z}(1 + d_K(z) - d_H(z))$ from $\text{cfan} \le 0$.

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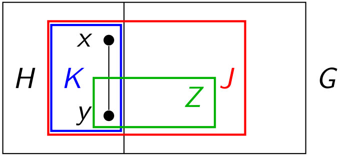

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- This gives the desired bound on $\sum_{z \in Z}(d_J(z) + 1 - \Delta(G))$.

Definition of B-Queues (Hoffman–Rodger 1988)

A B-queue of a simple graph B is a sequence of vertices $(u_1, \ldots, u_q)$ together with a sequence of vertex subsets $(S_0, S_1, \ldots, S_q)$ such that:

1. $S_0 = \emptyset$, and:
2. For all $i \in [q]$:
   - $S_i = N(u_i) \cup \{u_i\} \cup S_{i-1}$,
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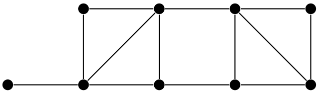
A B-queue is full if $S_q = V(B)$. **Example** (vertices of S_i are colored red):



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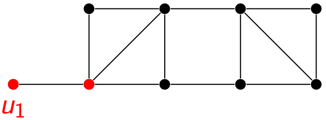

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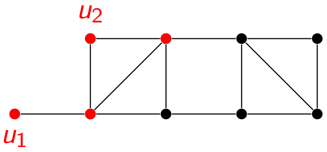

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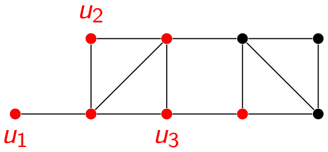

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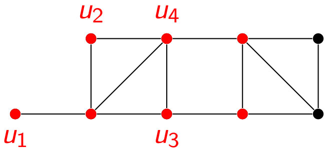

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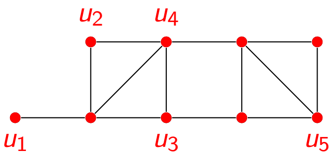
Definition of B-Queues (Hoffman–Rodger 1988)

A B-queue of a simple graph B is a sequence of vertices $(u_1, \ldots, u_q)$ together with a sequence of vertex subsets $(S_0, S_1, \ldots, S_q)$ such that:

1. $S_0 = \emptyset$, and:
2. For all $i \in [q]$:
   - $S_i = N(u_i) \cup \{u_i\} \cup S_{i-1}$,
   - $1 \leq |S_i \setminus S_{i-1}| \leq 2$,
   - $u_i \notin \{u_1, \ldots, u_{i-1}\}$, and
   - $|S_i \setminus (S_{i-1} \cup \{u_i\})| \leq 1$.

A B-queue is full if $S_q = V(B)$. **Example** (vertices of S_i are colored red):



Theorem (Hoffman–Rodger 1988)

If G is simple and its core admits a full B-queue, then $\chi'(G) = \Delta(G)$.

Full B-Queue $\implies$ $\mathrm{cfan}(B) \leq 0$

Theorem (McDonald–P.)

If B is a simple graph that admits a full B-queue, then $\mathrm{cfan}(B) \leq 0$.

Corollary (Hoffman–Rodger 1988)

Let G be a simple graph, and let H be its core. If H admits a full B-queue, then $\chi'(G) = \Delta(G)$.

Full B-Queue $\implies$ cfan$(B) \leq 0$

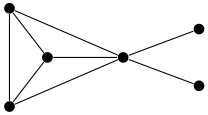
Theorem (McDonald–P.)

If B is a simple graph that admits a full B-queue, then cfan$(B) \leq 0$.

Corollary (Hoffman–Rodger 1988)

Let G be a simple graph, and let H be its core. If H admits a full B-queue, then $\chi'(G) = \Delta(G)$.

There are graphs with cfan$(B) \leq 0$ that do not admit a full B-queue, for example:



A Converse Result

Theorem (McDonald–P.)

Let G be a graph with core H. If $\text{cfan}(H) \leq 0$, then $\text{Fan}(G) \leq \Delta(G)$.

Theorem (McDonald–P.)

If H is a graph with $\text{cfan}(H) > 0$, then there exists a graph G with H as its core such that $\text{Fan}(G) > \Delta(G)$.

In other words, for a graph H, the following are equivalent:

1. $\text{cfan}(H) \leq 0$,
2. For every graph G with t-core H, $\text{Fan}(G) \leq \Delta(G)$.

What Happened to the Multigraphs?

Definition

The *t-core* of a multigraph G is the set of vertices with $d(v) + \mu(v) > \Delta(G) + t$.
(Ordinary "core" of a simple graph is just the 0-core.)

Theorem (McDonald–P.)

Let G be a multigraph with t-core H. If $\mathrm{cfan}(H) \leq t$, *then* $\chi'(G) \leq \mathrm{Fan}(G) \leq \Delta(G) + t$.

Goldberg–Seymour Conjecture

Definition

For a multigraph G, $w(G) = \max_{H \subset G} \left\lceil \frac{|E(H)|}{\lfloor |V(H)|/2 \rfloor} \right\rceil$.

Observe that always $\chi'(G) \geq w(G)$.

Theorem (Goldberg–Seymour Conjecture, Proof Announced by Chen–Jing–Zang)

For a multigraph G, $\chi'(G) \leq \max\{\Delta(G) + 1, w(G)\}$.

Corollary

For $t \geq 1$, $\chi'(G) \leq \Delta(G) + t$ if and only if $w(G) \leq \Delta(G) + t$.

This seems to largely obsolete other sufficient conditions for $\chi'(G) \leq \Delta(G) + t$ when $t \geq 1$.

FIN